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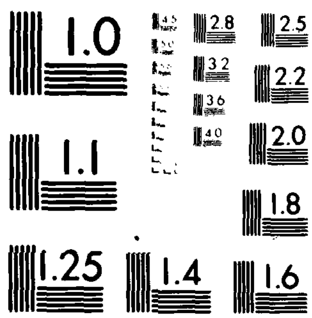

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AN EFFICIENT METHOD FOR PERFORMING
PARTIAL FRACTION EXPANSION

Research Report No. 80-11

by

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An escalation method for performing partial fraction expansions is presented for the case that the complete list of zeros of the denominator of the proper rational function is known. Expressions for the number of divisions and multiplications required are developed. The new method requires fewer such arithmetic operations than does the method of Henrici. A numerical example is provided.

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TABLE OF CONTENTS

Abstract

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Introduction. At times it is desired to express a rational function in terms of partial fractions. After completely factoring the denominator polynomial into linear factors it is conceptually easy to perform the expansion, but more efficient methods of carrying out the calculation are always welcome. Particular attention is directed to the situation where the linear factors of the denominator occur repeated. The partial fraction expansion technique that is presented here is efficient relative to the number of arithmetic operations required. It compares favorably in this regard with the method of Henrici [2].

The Escalation Process. Consider the proper rational function

$$\phi(x) = \frac{P(x)}{Q(x)(x-\xi_a)^{A-1}(x-\xi_b)^B} \tag{1}$$

where $\xi_a \neq \xi_b$, A and B are integers greater than zero, and $P(x)$ and $Q(x)$ are polynomials for which neither ξ_a nor ξ_b are zeros. In terms of partial fractions

$$\phi(x) = \sum_{i=1}^{A-1} \frac{C_{ai}}{(x-\xi_a)^i} + \sum_{i=1}^{B} \frac{C_{bi}}{(x-\xi_b)^i} + \chi(x) \tag{2}$$

Here and in other parts of this paper a summation is taken to vanish if its upper limit is less than its lower limit. A related function $\hat{\phi}(x)$ has the definition and partial fraction expansion given by

$$\hat{\phi}(x) = \frac{\phi(x)}{(x-\xi_a)} = \sum_{i=1}^{A} \frac{\hat{C}_{ai}}{(x-\xi_a)^i} + \sum_{i=1}^{B} \frac{\hat{C}_{bi}}{(x-\xi_b)^i} + \hat{\chi}(x) \tag{3}$$

In going from $\phi(x)$ to $\hat{\phi}(x)$ the power of $(x-\xi_a)$ in the denominator was increased by unity. For this reason those A partial fraction coefficients represented by $\hat{C}_{ai}$ are called the native coefficients of (3). All other partial fraction coefficients of (3) (the $\hat{C}_{bi}$ and those contained in $\hat{\chi}(x)$) are called <u>alien</u> coefficients. What follows will explore and exploit the conjecture that the partial fraction coefficients (both native and alien) of (3) may be computed from their counterparts in (2).

First consider the equation

$$(x-\xi_a)^{A-1}\phi(x) = (x-\xi_a)^A\hat{\phi}(x) \tag{4}$$

and the (A-2) equations obtained through repeated differentiation with respect to x (if A = 1, or 2, no differentiation is indicated). When x is set equal to ξ_a one finds that

$$\hat{C}_{ai} = C_{a,i-1} \quad , \quad i = 2, 3, \cdots, A \tag{5}$$

No information is gained concerning $\hat{C}_{a1}$.

Next consider

$$(x-\xi_b)^B\phi(x) = (x-\xi_b)^B(x-\xi_a)\hat{\phi}(x) \tag{6}$$

and the (B-1) equations obtained through successive differentiation. Upon setting $x=\xi_b$ one finds that

$$\hat{C}_{bB} = \frac{C_{bB}}{\xi_b-\xi_a} \tag{7}$$

and

$$\hat{C}_{bi} = \frac{C_{bi}-\hat{C}_{b,i+1}}{\xi_b-\xi_a} \quad , \quad i = B-1, B-2, \cdots, 2, 1 \tag{8}$$

A summary is now presented which makes use of signal flow graphs. The second row of coefficients in Figure 1 are the native coefficients of (3) while the first row are the corresponding coefficients of (2). In Figure 2 the analogous signal flow graph for the computation of a set of alien coefficients is presented. The procedure must be repeated for each different set of alien coefficients. For brevity take

$$f = \frac{1}{\xi_b-\xi_a} \tag{9}$$

Application of Escalation. Consider the proper rational function given by

$$F(s) = \frac{N(s)}{D(s)} \tag{10}$$

where

$$N(s) = \sum_{i=0}^{n-1} b_i s^i \tag{11}$$

and

$$D(s) = \sum_{i=0}^{n} a_i s^i \tag{12}$$

In (12) $a_n \neq 0$, but no such analogous restriction is implied in (11), except that not all the b_i are zero. The distinct zeros of D(s) are $\sigma_1, \sigma_2, \cdots, \sigma_q$ which occur with multiplicities of $M_1, M_2, \cdots, M_q$ respectively. The complete

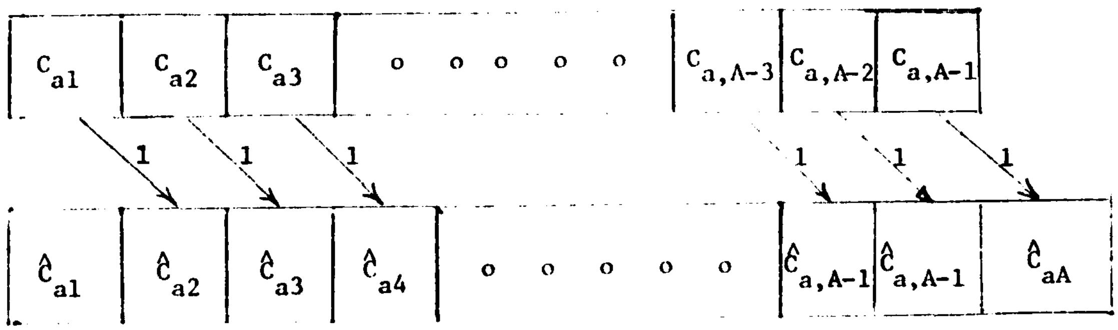

Figure 1. Formation of native coefficients.

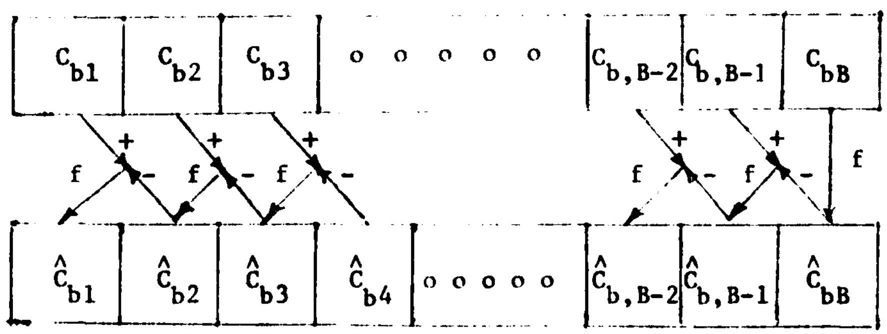

Figure 2. Formation of alien coefficients.

list of zeros of D(s) is $s_1, s_2, \cdots, s_n$ where it is convenient, but not necessary, to arrange the list such that the first M_1 items all equal σ_1, the next M_2 items all equal σ_2, and so forth until the final M_q items all equal σ_q. In factored form D(s) becomes

$$D(s) = a_n \prod_{j=1}^{n} (s-s_j) = a_n \prod_{j=1}^{q} (s-\sigma_j)^{M_j} \tag{13}$$

The numerator may be written as

$$N(s) = \beta_0 + \sum_{i=1}^{n-1} \beta_i \prod_{j=1}^{i} (s-s_j) \tag{14}$$

where the β coefficients are found by Horner's scheme. The details of this calculation will be illustrated later. Upon defining

$$G_n(s) = a_n F(s) \tag{15}$$

one obtains

$$G_n(s) = \frac{\beta_0 + \sum_{i=1}^{n-1} \beta_i \prod_{j=1}^{i} (s-s_j)}{\prod_{j=1}^{n} (s-s_j)} \tag{16}$$

This may be generalized to

$$G_r(s) = \frac{\beta_0 + \sum_{i=1}^{r-1} \beta_i \prod_{j=1}^{i} (s-s_j)}{\prod_{j=1}^{r} (s-s_j)}, \quad r=1, 2, \cdots, n \tag{17}$$

Algebraic manipulation then reveals

$$G_r(s) = \frac{G_{r-1}(s)}{(s-s_r)} + \frac{\beta_{r-1}}{(s-s_r)}, \quad r=1, 2, \cdots, n \tag{18}$$

provided that one defines

$$G_0(s) = 0 \tag{19}$$

All of the $G_r(s)$ defined by (17) are proper rational functions and hence have partial fraction expansions.

The immediate aim is to find the partial fraction expansion of $G_r(s)$ from the corresponding expansion of $G_{r-1}(s)$. If this can be accomplished, it can be repeated n times for r=1, 2, ..., n, ultimately yielding the expansion for $G_n(s)$. Then owing to (15) the expansion of F(s) is easily found.

The process of going from the partial fraction expansion of $G_{r-1}(s)$ to that of $G_r(s)$ may be broken into steps. First define

$$\hat{G}_r(s) = \frac{G_{r-1}(s)}{(s-s_r)} \quad , \quad r=1,\ 2,\ \cdots,\ n \tag{20}$$

and note that if the expansion of $G_{r-1}(s)$ is known that, with the exception of the first native coefficient, all of the partial fraction coefficients of $\hat{G}_r(s)$ may be found by invoking the processes shown in Figures 1 and 2. Then, according to (18), $\frac{\beta_{r-1}}{(s-s_r)}$ is added to the foregoing result. This addition causes the alteration of only one coefficient in $G_r(s)$ as compared to $\hat{G}_r(s)$. That altered coefficient is the same initial native coefficient for which the method of escalation sheds no light. Thus the escalation policy as given in Figures 1 and 2 is sufficient to transform the coefficients of $G_{r-1}(s)$ into all but one of the coefficients of $G_r(s)$. The remaining initial native coefficient of $G_r(s)$ may be found by using a theorem of Hazony [1].

Hazony's theorem states that the sum of the residues of any rational function whose denominator degree exceeds its numerator degree by two or more is zero. It has been noted that $G_{r-1}(s)$ is a proper rational function. This means that the denominator degree exceeds the numerator degree by at least unity. Since $\hat{G}_r(s)$ is formed according to (20) it is clear that $\hat{G}_r(s)$ fulfills the requirements of Hazony's theorem. Thus the sum of the residues of $\hat{G}_r(s)$ is zero, and owing to (18), the residue sum of $G_r(s)$ is β_{r-1}. Since the unknown initial native coefficient of $G_r(s)$ is a residue, its value may be found by noting that those partial fraction coefficients of $G_r(s)$ which may be identified as residues sum to β_{r-1}.

Illustration of the Method. For a more concrete presentation, attention is directed to the finding of the partial fraction expansion of

$$F(s) = \frac{1+2s+3s^2+4s^3+5s^4+6s^5}{24-104s+182s^2-164s^3+80s^4-20s^5+2s^6} \tag{21}$$

In a separate (and by no means trivial) calculation it may be found that: $\sigma_1=1$, $M_1=3$; $\sigma_2=2$, $M_2=2$; $\sigma_3=3$, $M_3=1$. This means that the complete list of the zeros of D(s) is: 1, 1, 1, 2, 2, 3. The β values may be found by Horner's scheme, which involves the repeated use of synthetic division using the complete list of zeros of D(s) as divisors. The initial dividend is the coefficients of N(s) and the remainders are the β coefficients. The process in abbreviated form is now shown.

$$\begin{array}{l|llllll}
\underline{1} & [6\ , & 5\ , & 4\ , & 3\ , & 2\ , & 1] \\
\underline{1} & [6\ , & 11\ , & 15\ , & 18\ , & 20] & 21 \\
\underline{1} & [6\ , & 17\ , & 32\ , & 50] & 70 & \\
\underline{2} & [6\ , & 23\ , & 55] & 105 & & \\
\underline{2} & [6\ , & 35] & 125 & & & \\
\underline{3} & [6\ ] & 47 & & & & \\
 & 6 & & & & &
\end{array}$$

It may be verified that

$$N(s) = 21 + 70(s-1) + 105(s-1)^2 + 125(s-1)^3 + 47(s-1)^3(s-2) + 6(s-1)^3(s-2)^2 \tag{22}$$

Next, use is made of the tableau given in Figure 3.

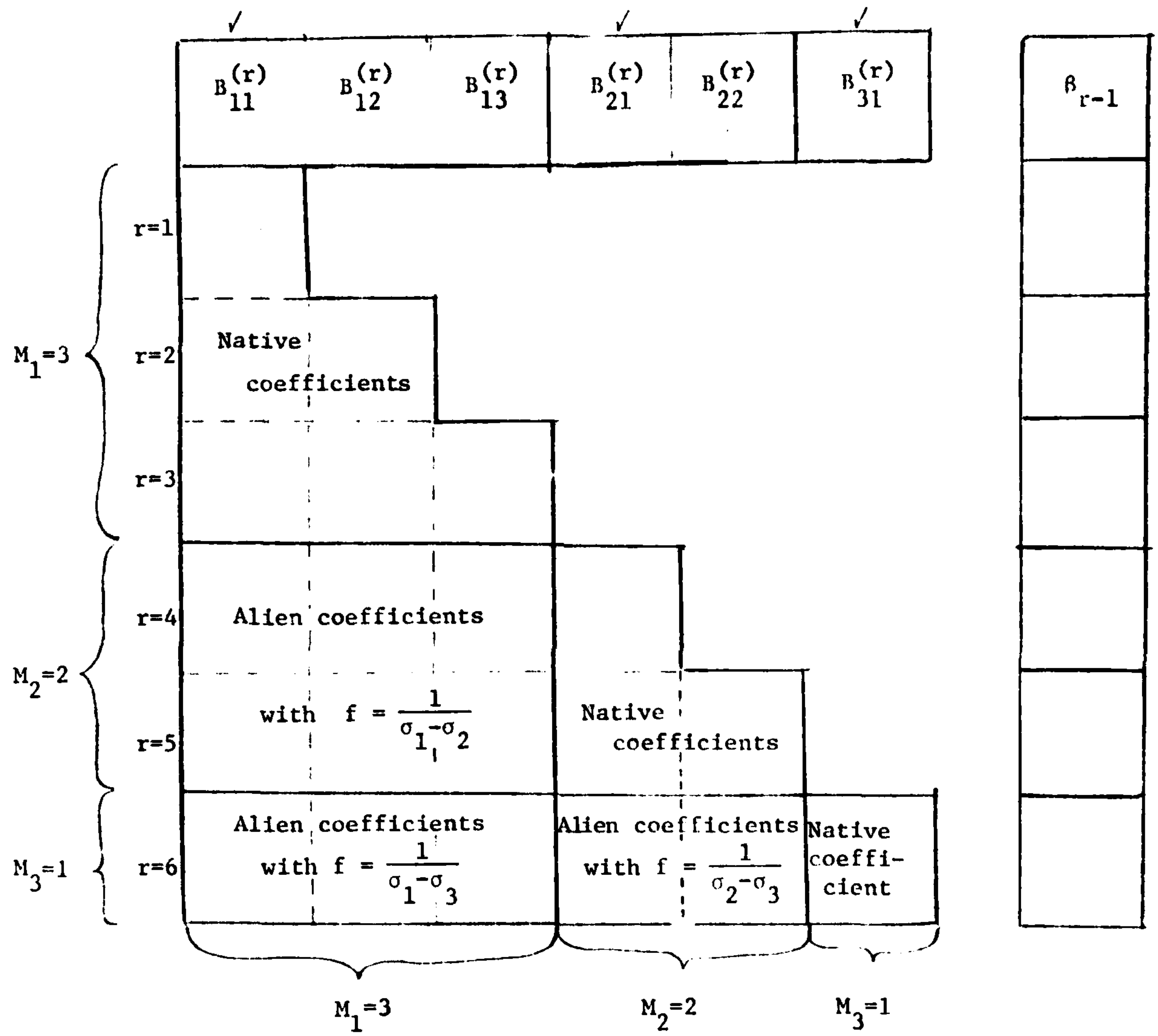

Figure 3. Tableau for performing escalation.

Each row of the main body of the tableau contains the partial fraction coefficients of $G_r(s)$ defined by

$$G_r(s) = \sum_{i=1}^{q} \sum_{j=1}^{L_i} \frac{B_{ij}^{(r)}}{(s-\sigma_i)^j} \quad , \quad r=1, 2, \cdots, n \tag{23}$$

where the L_i are the appropriate non-negative integers such that $r = \sum_{i=1}^{q} L_i$, where none of the L_i exceeds M_i. The last row of the tableau contains the coefficients for $G_n(s)$, which are easily converted into the coefficients of $F(s)$ through division by a_n.

Those regions in the tableau labled "native coefficients" are filled in from the row above by using the process shown in Figure 1 and by demanding that the row sum of those coefficients that are identified as residues (they bear check marks) sum to give β_{r-1}. The "alien coefficients" are filled in from the row above by using the procedure given in Figure 2. The calculation progresses from the top row to the bottom row.

Omitting the details of the calculation, the entries in the tableau are:

21						21
70	21					70
105	70	21				105
-196	-91	-21	321			125
308	112	21	-261	321		47
$-\frac{1477}{8}$	$-\frac{245}{4}$	$-\frac{21}{2}$	-60	-321	$\frac{2005}{8}$	6

Division of the last row coefficients by $a_n(=2)$ gives the partial fraction expansion of $F(s)$.

$$F(s) = \frac{-\frac{1477}{16}}{(s-1)} + \frac{-\frac{245}{8}}{(s-1)^2} + \frac{-\frac{21}{4}}{(s-1)^3}$$

$$+ \frac{-30}{(s-2)} + \frac{-\frac{321}{2}}{(s-2)^2} + \frac{\frac{2005}{16}}{(s-3)} \tag{24}$$

Operations count. In order to compare the efficiency of a given method it is useful to have expressions for the number of multiplications and divisions involved, assuming that additions and subtractions are less troublesome. Let m (<n) be the degree of the numerator. In order to find the β coefficients $\frac{(m)(m+1)}{2}$ multiplications are required when Horner's scheme is used. Referring to Figure 3 it is seen that the native coefficients do not require any multiplications or divisions. This assumes that the multiplications by unity indicated in Figure 1 are too trivial to be counted. Each alien coefficient could be computed with only one division. This assumes that one could divide by $(\sigma_1-\sigma_2)$ rather than multiplying by $\frac{1}{(\sigma_1-\sigma_2)}$ as indicated in Figure 2. There are $\frac{n^2}{2} - \frac{S}{2}$ alien coefficients and hence that many divisions, where

$$S = M_1^2 + M_2^2 + \cdots + M_q^2 \tag{25}$$

Finally, there are n divisions by a_n, although a_n frequently equals unity. The sum of all the above operations is

$$\frac{n^2}{2} + n - \frac{S}{2} + \frac{(m)(m+1)}{2}$$

The last term may range from zero when m = 0, up to $\frac{(n)(n-1)}{2}$ when m = n - 1. Hence the operations count varies from a low value of

$$\frac{n^2}{2} + n - \frac{S}{2}$$

to a high value of

$$n^2 + \frac{n}{2} - \frac{S}{2}$$

Henrici [2] gives the operations count for his method as being less than

$$2n^2 + S.$$

The present author reanalyzed the method and under the assumption that $(n+1) \geq 2M_i$, $i = 1, 2, \cdots, q$, found that the low value (for m=0) is

$$2n^2 + n - 2S$$

and the high value (for $m = n - 1$) is

$$3n^2 + \frac{1}{2}n - \frac{5}{2}S$$

Suppose each zero were to occur with the same multiplicity M, then

$$S = nM$$

It is now seen that when n is large, the operations count for escalation is less than one half of the count for Henrici's method.

References

[1] Hazony, D., and Riley, J., "Evaluating Residues and Coefficients of High Order Poles," IRE Trans. Auto. Cont., AC-4, (1959).

[2] Henrici, P., Applied and Computational Complex Analysis, Volume 1, John Wiley and Sons, (1974).